

# GRAVITATING OPPOSITES ATTRACT

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ABSTRACT. Generalizing previous work by two of us, we prove the non-existence of certain stationary configurations in General Relativity having a spatial reflection symmetry across a non-compact surface disjoint from the matter region. Our results cover cases such that of two symmetrically arranged rotating bodies with anti-aligned spins in  $n+1$  ( $n \geq 3$ ) dimensions, or two symmetrically arranged static bodies with opposite charges in 3+1 dimensions. They also cover certain symmetric configurations in (3+1)-dimensional gravity coupled to a collection of scalars and abelian vector fields, such as arise in supergravity and Kaluza-Klein models. We also treat the bosonic sector of simple supergravity in 4+1 dimensions.

Keywords: stationary  $n$ -body problem

## 1. INTRODUCTION

The title of this paper is an allusion to that of one by Aharonov, Casher, Coleman, and Nussinov [ACS] who give general arguments why solitons in flat space, such as Yang-Mills monopoles, with the opposite charges always attract. They showed, for example, that for a monopole-antimonopole configuration in Yang-Mills-Higgs theory, one may always lower the energy by moving the monopole-anti-monopoles together. This does not mean that the forces between them are necessarily attractive, and indeed, Taubes [T] has shown that there exist static monopole-antimonopole solutions. However these solutions are unstable.

The forces considered by [ACS] were classical, but in a subsequent paper Kenneth and Klich [KK] showed that a similar property holds for quantum field-theoretic Casimir forces [KK]. This was followed by a paper of Bachas [BA] which considerably extended this result and related this behaviour to the fundamental property of reflection positivity in quantum field theory.

It is natural therefore to ask whether the statement continues to hold in the presence of gravity, according to General Relativity. In the case of purely gravitational forces between static bodies this question has been addressed in a recent paper by two of us [BSc]. It was shown that a bi-partite system of gravitating bodies <sup>1</sup> satisfying the conditions for which the positive mass theorem holds, the two parts being separated by a submanifold  $S$  immersed in vacuum

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<sup>1</sup>The word “bodies” should be taken to include ordinary bodies with compactly supported energy momentum tensors and black holes with regular event horizons

and which is totally geodesic with respect to the the spatial metric, cannot rest in static equilibrium.

The theorem proved in [BSc] holds in the case that the bodies exert no long range forces other than gravity upon each other, in other words outside the bodies the Einstein vacuum equations hold. We turn now to the case when the bodies carry electric charges. In the static case we expect that there can be no static equilibrium between oppositely charged bodies and a simple modification of the method used in [BSc] allows us to prove this (**Theorem 3.1**). Previous results, e.g [G80] assumed axisymmetry. No such assumption is necessary for our result.

It is well known that rotating gravitating bodies, including black holes, exert spin-spin forces [WA] which may be repulsive or attractive depending upon the orientations of their angular momenta. For example an axisymmetric system of two bodies for which their angular momenta are aligned in the *same* direction along the line of centres experiences a repulsion, while if the two angular momenta are *anti-aligned* they experience an attraction [WA]. In the latter case (which we think of as opposites in a sense to be made more precise later) we do not expect an equilibrium to be possible. The question of whether the gravitational spin-spin interaction can ever overcome the gravitational attraction is a long-standing one which has been extensively studied for axisymmetric stationary metrics [WE] and recently been resolved [NH] (for a discussion in 4+1 dimensions see [HRZ]). As far as we are aware, there are no rigorous results for stationary system which are not axisymmetric. One of the principal aims of the present paper is to generalize the argument given in [BSc] to prove in **Theorem 4.1** without assuming axisymmetry, that in the case of oppositely aligned spins no such equilibrium is possible.

To establish the result for rotating bodies it proves advantageous to modify slightly the method of [BSc] by conformally rescaling the spatial metric. Not only does this streamline some of the calculations but it allows for an interesting physical interpretation of putative equilibria in terms of the balance of stresses in the system, completely analogous to the Newtonian treatment of equilibria in terms of a gravitational stress tensor  $t_{ij}^U$  constructed from the Newtonian potential  $U$  and a matter stress tensor  $t_{ij}^{\text{mat}}$ . In the case of stationary spacetimes there is a contribution to the stress tensor from the twist  $\omega$  of the Killing vector. In the case of Einstein-Maxwell theory, the matter stress tensor is constructed from the electro-static and magneto-static potentials  $\phi$  and  $\chi$ .

This formalism allows an extension to cover much more complicated matter systems. In particular it covers the systems discussed in [G82, BMG] which arise in supergravity, Kaluza-Klein and String theories.

## 2. NEWTONIAN CONSIDERATIONS

One of the first things we are taught in physics is that like charges repel and unlike charges attract. By “charges” is usually meant electric charges and for example, one could consider the force between a pair of apparently identical sources. If the two sources have the same electric charge, there will be a repulsive force and if they have the opposite electric charge there will be an equal but

opposite attraction. The same holds for magnetism, and indeed it seems that historically what is usually called Coulomb's law was first established for magnetic poles by John Michell [WH], inventor of the torsion balance subsequently used by Cavendish to measure Newton's constant  $G$ , and of the concept of a black hole. Michell would have been aware that because gravity is always attractive an equilibrium between two equal masses with opposite charges is never possible. If the charges are the same, then of course by special choice of the charges and masses an equilibrium may be possible. More generally one may consider a bi-partite system in which two disjoint sets of charges and masses, or a continuous distribution of matter and electric are symmetrically disposed with respect to a plane  $S$  such that the charges on one side are the opposite of the charges on the other side. The net gravitational force between them is clearly attractive<sup>2</sup> and the net electric force between them will also be attractive and again no equilibrium is possible. A formal justification of this might proceed by considering the electrostatic and Newtonian gravitational stress tensors

$$t_{ij}^{\phi} = (D_i \phi)(D_j \phi) - \frac{1}{2} \delta_{ij} (D\phi)^2 \quad (2.1)$$

$$t_{ij}^U = -\frac{1}{4\pi G} (D_i U)(D_j U) + \frac{1}{8\pi G} \delta_{ij} (DU)^2 \quad (2.2)$$

where  $\phi$  and  $U$  are the electrostatic and gravitational potentials.

Since the electric field lines which start on the positive charges must end on the negative charges, and symmetry dictates that  $E_i = -\partial_i \phi$  is orthogonal to  $S$  there is a net electric flux crossing  $S$  and therefore a net attractive electric force

$$F_i^{\phi} = \int_S t_{ij}^{\phi} d\sigma_j \quad (2.3)$$

The gravitational potential must by contrast be constant on  $S$  but because of the opposite sign in (2.2) compared with (2.1) there is again a net attractive force between the systems and no equilibrium is possible.

A general necessary condition for equilibrium in flat spacetime

$$\int_S t_{ij} d\sigma_j = 0, \quad (2.4)$$

where  $t_{ij}$  is the total stress tensor. In some cases it may be used to rule out the existence of certain static equilibria. For example one can show that (2.4) can not be satisfied on any plane separating the bodies. Conversely, where such a plane does not exist, one can in fact construct 2-body configurations [BS], and analogues have been found in GR in [AS]. These equilibrium configurations will in general not be stable. Let us remark that nowhere in this paper do we touch the issue of stability. In what follows we shall generalize condition (2.4) to incorporate the effects of General Relativity.

In flat space the necessary condition (2.4) must hold in particular for the Yang-Mills-Higgs equations considered in [ACS, T]. If the Higgs potential vanishes and the fields satisfy the first order Bogomolny equations, then the total

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<sup>2</sup>provided of course that the gravitational masses are taken to be positive, which is not mandatory in Newtonian theory [F]

stress tensor  $t_{ij}$  is known to vanish pointwise. This is consistent with the existence of the well known static multi-monopole solutions of the first order equations. We shall show shortly that a similar statement holds for static charged multi-black hole solutions in general relativity. Taubes's monopole-antimonopole solutions however satisfy the second order equations but not the first order equations. It is not obvious to us whether (2.4) is satisfied because the stresses vanish pointwise on  $S$ , or because a cancellation in the integral.

Of course the notion of force does not make much sense in General Relativity, but the notion of a stress tensor for the electromagnetic field certainly does and, as we shall see shortly, as does, if the system is static, the idea a gravitational stress tensor. This will allow us to implement the ideas above in a general relativistic context and to extend the recent proof [BSc] on the non-existence of static n-body configurations in pure gravity to the electrostatic case.

Something which was not envisaged by Newton, although it has a correspondence in Newtonian theory is the frame-dragging forces exerted by rotating bodies. We shall show, in effect by introducing a suitable stress tensor for magneto-gravitational forces, how the results of [BSc] can be extended to stationary bodies.

For a general discussion of the fact that forces mediated by fields of even(odd) spin are attractive(repulsive) see [D].

### 3. ELECTROSTATIC CASE

We begin, for the sake of simplicity, by giving the simplest generalization of the argument of [BSc].

If we write the static metric as

$$ds^2 = -V^2 dt^2 + g_{ij} dx^i dx^j \quad A = \phi dt, \quad \kappa = 8\pi G \quad (3.1)$$

$$R_{ij} = \frac{1}{V} D_i D_j V - \frac{\kappa}{V^2} \left[ (D_i \phi)(D_j \phi) - \frac{1}{2} g_{ij} (D\phi)^2 \right], \quad (3.2)$$

$$V \Delta V = \frac{\kappa}{2} (D\phi)^2, \quad (3.3)$$

$$V \Delta \phi = g^{ij} (D_i \phi)(D_j V). \quad (3.4)$$

As a check, note that we obtain the standard Hamiltonian constraint

$$R = \frac{\kappa}{V^2} (D\phi)^2, \quad (3.5)$$

as expected

Following [BSc] we assume that the space of orbits,  $N$ , of the timelike Killing field admits a totally geodesic surface  $S$  with Gauss-curvature  $K$ , then

$$R_\alpha{}^\alpha = 2K + R_{nn}, \quad (3.6)$$

where the first term on the left is the tangential trace with respect to the metric on  $S$  and  $R_{nn}$  is defined by  $R_{nn} = R_{ij} n^i n^j$ , with  $n^i$  being the unit normal of  $S$ . Taking the tangential trace of (3.2) gives

$$R_\alpha{}^\alpha = \frac{1}{V} \Delta_S V + \frac{\kappa}{V^2} (D_n \phi)^2 \quad (3.7)$$

Thus we arrive at

$$K = \frac{1}{V} \Delta_S V + \frac{\kappa}{2V^2} \left[ (D_n \phi)^2 - (D_\alpha \phi)(D^\alpha \phi) \right], \quad (3.8)$$

If we assume that  $S$  is an iso-potential of  $\phi$  we have  $D_\alpha \phi = 0$  and we may proceed as in [BSc].

**3.1. Unlike charges.** Our assumptions would hold if we had a static system of charged bodies invariant under an isometric action of  $\mathbb{Z}_2$  which stabilizes  $S$  pointwise and under which the electric field is *odd*

$$\mathbb{Z}_2 : D\phi \rightarrow -D\phi. \quad (3.9)$$

This is the analogue of the situation considered in [ACS] for solitons and in [KK, BA] for Casimir forces. Thus we obtain the

**Theorem 3.1.** *Assume that  $(N, g)$  is static electrovacuum outside a compact set and has  $R \geq 0$  everywhere. Suppose there is a properly embedded, noncompact, totally geodesic surface  $S$  such that  $g$  is static electrovacuum in a neighborhood of  $S$  and that the pull-back-to- $S$  of  $D_i \phi$  is zero. It follows that  $(N, g)$  is isometric to the Euclidean space  $\mathbb{R}^3$  and  $(M, ds^2) = (\mathbb{R} \times N, -V^2 dt^2 + g)$  is Minkowski space.*

For the proof we refer the reader to the generalization of this result to Einstein-Maxwell-Dilaton theory in Sect.5.

**3.2. Like Charges.** In this case we assume that electric field is *even* under the  $\mathbb{Z}_2$  action

$$\mathbb{Z}_2 : D\phi \rightarrow D\phi. \quad (3.10)$$

We also have, in both even and odd cases

$$D_n V = 0, \quad (3.11)$$

Now we know of one example, the Majumdar-Papapetrou solutions [S] for which equilibrium is possible. This is if

$$D\phi = \pm \sqrt{\frac{2}{\kappa}} DV. \quad (3.12)$$

$$g_{ij} = V^{-2} \delta_{ij}. \quad (3.13)$$

and in fact  $V^{-1}$  is harmonic wrt to the flat metric  $\delta_{ij}$ . In that case  $S$  is conformally flat and hence  $K \neq 0$ . However

$$\int_S K = \int_S \frac{1}{V^2} \left[ (D_\alpha V)(D^\alpha V) + \frac{\kappa}{2} (D_n \phi)^2 - \frac{\kappa}{2} (D_\alpha \phi)(D^\alpha \phi) \right] = 0, \quad (3.14)$$

as required.

A complete treatment of the necessary and sufficient conditions for equilibrium of charged gravitating objects with the same sign in General Relativity is not yet completely available (see [B]). The most recent results in the axisymmetric case can be found in [M].

4. STATIONARY CASE IN  $n + 1$  DIMENSIONS

We assume  $(M, ds^2)$  has a timelike Killing vector  $\xi^\mu$  with complete orbits. Then  $ds^2$  can be written as

$$ds^2 = -e^{2U} (dt + \psi_i dx^i)^2 + e^{-\frac{2U}{n-2}} h_{ij} dx^i dx^j \quad (4.1)$$

with  $h$  being a Riemannian metric on  $N$ , the quotient space under the action of  $\xi = \partial_t$ . We refer to  $U$  as the gravitational potential and define  $1/2$  the curvature of the Sagnac connection  $\psi$  by

$$\omega = \frac{1}{2} d\psi, \quad \omega_{ij} = \partial_{[i} \psi_{j]} \quad (4.2)$$

We only need the gravitational part of the action. There is the identity

$$R\sqrt{-g} = \left[ \frac{2}{n-2} \Delta_h U + \mathcal{R} - \frac{n-1}{n-2} (DU)^2 + e^{2U \frac{n-1}{n-2}} \omega_{kl} \omega^{kl} \right] \sqrt{h}, \quad (4.3)$$

where  $\mathcal{R}$  is the Ricci scalar of  $h_{ij}$ . Hence, modulo a surface term, the reduced gravitational action is given by

$$S = \int \left[ \mathcal{R} - \frac{n-1}{n-2} (DU)^2 + e^{2U \frac{n-1}{n-2}} \omega_{kl} \omega^{kl} \right] \sqrt{h} d^3x \quad (4.4)$$

and the vacuum field equations turn out to be ( $\mathcal{G}_{ij} = \mathcal{R}_{ij} - \frac{1}{2} h_{ij} \mathcal{R}$ )

$$\mathcal{G}_{ij} + 8\pi G (t_{ij}^U + t_{ij}^\omega) = 0 \quad (4.5a)$$

$$\Delta_h U + e^{2U \frac{n-1}{n-2}} \omega_{kl} \omega^{kl} = 0 \quad (4.5b)$$

$$D^j \left( e^{2U \frac{n-1}{n-2}} \omega_{ij} \right) = 0 \quad (4.5c)$$

In (4.5a) we are using the definitions

$$8\pi G t_{ij}^U = \frac{n-1}{n-2} [-(D_i U)(D_j U) + \frac{1}{2} h_{ij} (DU)^2] \quad (4.6)$$

and

$$8\pi G t_{ij}^\omega = 2 e^{2U \frac{n-1}{n-2}} [\omega_{ik} \omega_j^k - \frac{1}{4} h_{ij} \omega_{kl} \omega^{kl}] \quad (4.7)$$

(Using  $d\omega = 0$ , Eq.(4.5c) follows from Eq.'s (4.5a,b).) We consider asymptotically flat solutions of (4.5). They can be shown (see e.g. [MP]) to have the form

$$h_{ij} = \delta_{ij} + o^\infty \left( \frac{1}{r^{n-2}} \right) \quad (4.8a)$$

$$U = -\frac{8\pi}{(n-1)A_{n-1}} \frac{m}{r^{n-2}} + o^\infty \left( \frac{1}{r^{n-2}} \right) \quad (4.8b)$$

$$\omega_{ij} = -\frac{4\pi}{A_{n-1}} \frac{L_{ij} + n a_{[i} L_{j]k} a^k}{r^n} + o^\infty \left( \frac{1}{r^n} \right) \quad (4.8c)$$

Here  $a^i = \frac{x^i}{r}$  and  $A_n$  is the area of  $\mathbb{S}^n$ . Furthermore the constants  $m$  and  $L_{ij} = L_{[ij]}$  are respectively the ADM mass and the (mass-centered) spin tensor of the configuration. When  $m = 0$ , the ADM energy of the initial data set induced on  $t = \text{const}$  is also zero. Then, when the source satisfies the dominant

energy condition, the positive energy theorem implies that spacetime is flat<sup>3</sup>. When there is more than one asymptotically flat end, all the statements above hold separately w.r. to any such end.

Suppose  $(N, h)$  admits a reflection symmetry  $\Psi$  across an  $(n-1)$  - dimensional surface  $S$ , which is disjoint from the matter region. There are two ways this can be lifted to an isometry of  $(M, g)$ . One is that if, in addition to  $\Psi^*h = h$ , there holds  $\Psi^*U = U$  and  $\Psi^*\omega = \omega$ . Then, provided that  $N$  is simply connected, there is a reflection of  $(M, g)$  which preserves  $\xi$ . (This reflection preserves the timelike hypersurface  $\Sigma$  pointwise consisting of the orbits of  $S$  under the flow of  $\xi$ .) The other possibility is that if  $\Psi^*U = U$  and  $\Psi^*\omega = -\omega$ , in which case there is a reflection of  $(M, g)$  which maps  $\xi$  to  $-\xi$ . In both cases we have that  $D_n U|_S = 0$ , where  $n^i$  is a normal of  $S$  and that  $\omega_{ij}n^j|_S = 0$  in the first case and that the pull-back-to- $S$  of  $\omega$  be zero in the second case.

When the totally geodesic hypersurface  $S$  is properly embedded, non-compact and closed, by a straightforward extension of Propositions 2.1. and 3.1. in [BSc],  $S$  can be shown to be asymptotic to a disjoint union of a finite number of hyperplanes at infinity. Let some such asymptotic end of  $S$  be given by  $x^n = 0$ . It then follows from (4.8) that the spin tensor has to satisfy  $L_{\alpha n} = 0$  in the first case and  $L_{\alpha\beta} = 0$  in the second, where  $\alpha, \beta = 1, \dots, n-1$ .

Now the stress energy tensor of the matter source can be viewed as a superposition of two stress energy tensors  $T_{\mu\nu}^{\text{mat}}$  and  $T_{\mu\nu}'^{\text{mat}}$  which are images of each other under the map  $\Psi$  (i.e.  $(\Psi^*T)_{ij}^{\text{mat}} = T_{ij}'^{\text{mat}}$ ,  $\Psi^*\rho = \rho'$ ,  $(\Psi^*j)_i = \pm j'_i$  in obvious notation). Then, heuristically, we have that  $L_{ij} = L_{ij} + L'_{ij}$  with  $L, L'$  being the 'spins of the individual configurations'. Thus there holds  $L_{\alpha\beta} = L'_{\alpha\beta}$  and  $L_{\alpha n} = -L'_{\alpha n}$  in the first case and  $L_{\alpha\beta} = -L'_{\alpha\beta}$  and  $L_{\alpha n} = L'_{\alpha n}$  in the second case. We now argue that these two cases correspond to a repulsive (resp. attractive) spin-spin force between the two configurations. To see this we use, following [WA], the Mathisson-Fock-Papapetrou expression for the force  $F_i$  on a particle at rest, i.e.

$$F_i = -\frac{1}{2}L_{\text{test}}^{jk}R_{jki0} \quad \text{large } r \quad (4.9)$$

Here  $L_{\text{test}}^{ij}$  is the spin tensor of a test particle. Now from the Killing identity

$$\nabla_\mu \nabla_\nu \xi_\lambda = -R_{\nu\lambda\mu\rho} \xi^\rho \quad (4.10)$$

applied to the Killing vector  $\xi^\mu \partial_\mu = \partial_t$  we find, using (4.1, 4.2), that

$$\partial_i \omega_{jk} = R_{jki0} \quad (4.11)$$

to leading order. It follows that the spin-spin force is attractive (resp. repulsive) to leading order in  $1/r$ , whenever the quantity  $\omega_{ij}L_{\text{test}}^{ij}$  is negative (resp. positive). This, in turn, is the same as the quantity

$$L_{ij}^{\text{source}} L_{\text{test}}^{ij} - n L_{ij}^{\text{source}} L_{\text{test}k}^i n^j n^k = L_{\alpha\beta}^{\text{source}} L_{\text{test}}^{\alpha\beta} - (n-2) L_{\alpha n}^{\text{source}} L_{\text{test}}^{\alpha n} \quad (4.12)$$

being negative (resp. positive), where  $n^i$  is the unit vector pointing from the source to the test particle. So, indeed, this is consistent with the force between

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<sup>3</sup>When  $n = 3$  one can also allow for the presence of horizons, see [GHP],[BC].



rotating black holes being attractive when  $L_{\alpha\beta} = -L'_{\alpha\beta}$  and  $L_{\alpha n} = L'_{\alpha n}$  and repulsive when  $L_{\alpha\beta} = L'_{\alpha\beta}$  and  $L_{\alpha n} = -L'_{\alpha n}$ .

For example in 3+1 dimensions, at large separation  $r = |\mathbf{r}|$ , the mutual potential energy of two spinning bodies with angular momentum  $\mathbf{J}$  and  $\mathbf{J}'$  ( $J_i = \epsilon_{ijk} S^{jk}$ ) is given by [WA]

$$\frac{G}{c^2 r^3} \left[ 3(\mathbf{n} \cdot \mathbf{J})(\mathbf{n} \cdot \mathbf{J}') - r^2(\mathbf{J} \cdot \mathbf{J}') \right]. \quad (4.13)$$

This gives an attractive force if  $\mathbf{J} = -\mathbf{J}'$  and a repulsive force if  $\mathbf{J} = +\mathbf{J}'$ . Our results are consistent with this. They are also consistent with the behavior of explicit exact *double Kerr* solutions obtained using solution-generating techniques. These exhibit conical singularities along the axis between the two sources which is interpreted as a strut or rod in tension which holds the two black holes apart. For a recent detailed discussions see [HR, NH, CHR].

We now state as the main result of this section the

**Theorem 4.1.** *Let  $(N, h)$  have a hypersurface  $S$  disjoint from the matter region, which is non-compact, closed and totally geodesic w.r. to the unrescaled metric  $e^{-\frac{2U}{n-2}} h_{ij}$ . (Our previous remarks on possible 'liftings' of isometries from  $(N, h)$  to  $(M, ds^2)$  are equally valid when applied to  $(N, e^{-\frac{2U}{n-2}} h)$ .) Suppose in addition that the pull-back-to- $S$  of  $\omega$  is zero. Then spacetime is flat.*

**Remarks:** By the discussion above, this result covers the 'good' case, in which there is no chance for the spin-spin force to balance the gravitational attraction. Note also that we do not assume  $D_n U|_S = 0$ .

**Proof:** Contracting (4.5a) with  $n^i n^j$ , with  $n^i$  being the unit normal to  $S$  in  $(N, h)$  we find

$$\mathcal{G}_{nn} = \frac{n-1}{n-2} \left[ \frac{1}{2} (D_n U)^2 - \frac{1}{2} (D_\alpha U)(D^\alpha U) \right] - e^{2U \frac{n-1}{n-2}} \omega_{\alpha n} \omega^\alpha_n. \quad (4.14)$$

Here  $D_\alpha$  is the intrinsic derivative on  $S$ . In (4.14) we have used that  $\omega_{\alpha\beta}$  is zero. Now, from the Gauss equation,

$$\mathcal{R} = -2\mathcal{G}_{nn} + (\text{tr } k)^2 - \text{tr}(k^2), \quad (4.15)$$

where  $\mathcal{R}$  is the Ricci scalar of  $S$  and  $k$  its extrinsic curvature. Since  $S$  is totally geodesic w.r. to  $e^{-\frac{2U}{n-2}} h$ , we have that

$$k_{\alpha\beta} = \frac{1}{n-2} q_{\alpha\beta} D_n U \quad (4.16)$$

where  $q$  is the intrinsic metric on  $S$  induced from  $h$ . Inserting (4.16, 4.14) into Eq.(4.15), the terms involving  $D_n U$  cancel so that finally

$$\mathcal{R} = \frac{n-1}{n-2} (D_\alpha U)(D^\alpha U) + 2e^{2U \frac{n-1}{n-2}} \omega_{\alpha n} \omega^\alpha_n. \quad (4.17)$$

In particular  $\mathcal{R}$  is non-negative. By virtue of the 'asymptotically planar' nature of  $S$  (see [BSc] for details) the metric  $q$  on  $S$  tends to the Euclidean metric on  $S$  as fast as  $h$  tends to the flat metric on  $N$ , so  $S$  has zero ADM mass. Now for  $n > 3$  we can apply the positive energy theorem <sup>4</sup> to  $S$  yielding that  $S$  is

<sup>4</sup>When  $n = 4$  this is the standard positive energy theorem in 3 dimensions [SY]. For the positive energy theorem in general dimensions see [SC].



flat  $\mathbb{R}^{n-1}$ , in particular  $\mathcal{R}$  is zero, whence  $U|_S$  is zero. The same conclusions are reached for  $n = 3$ , by using the Gauss-Bonnet theorem. Consequently  $m = 0$  and spacetime is flat, again by the general positive energy theorem.

## 5. A GENERAL SIGMA-MODEL FORMALISM

In [BMG] a general formalism was developed to cover stationary solutions of Einstein's equations in 3+1 dimensions coupled to the matter sectors of various supergravity or Kaluza-Klein type theories. The models typically contain  $n_s$  scalar fields and  $n_v$  abelian vector fields. The same metric ansatz was made, and after some work all vector fields were swapped for  $n_v$  generalised electrostatic potentials and  $n_v$  magneto-static potentials. All such fields could be combined into a collection of scalar fields  $\Phi^A$ , where  $A = 1, 2, \dots, 2+n_s+2n_v$ . The fields  $\Phi^A$  include the Newtonian potential  $U$  and the Sagnac curvature  $\omega$ , whose precise definition depends on the matter fields considered and may be regarded as providing a map into some target space  $M_\Phi$  with, a metric  $G_{AB}(\Phi^C)$  with signature  $(2+n_s, 2n_v)$ . In many cases  $M_\Phi, G_{AB}(\Phi)$  is a pseudo-riemannian symmetric space  $G/H$ . The effective Lagrangian is

$$\int [\mathcal{R} - 2h^{ij}G_{AB}(\partial_i\Phi^A)(\partial_j\Phi^B)] \sqrt{h} d^3x \quad (5.1)$$

Thus the equations of motion reduce to the statement that the map  $\Phi^A(x)$  be harmonic and that

$$\mathcal{R}_{ij} - \frac{1}{2}\mathcal{R}h_{ij} = 8\pi G t_{ij}^\Phi \quad (5.2)$$

where

$$4\pi G t_{ij}^\Phi = G_{AB}(\partial_i\Phi^A)(\partial_j\Phi^B) - \frac{1}{2}h_{ij}h^{kl}G_{AB}(\partial_k\Phi^A)(\partial_l\Phi^B). \quad (5.3)$$

Like in (4.14) we see that for any chance of an equilibrium the quantity  $t_{nn}^\Phi$  given by

$$-8\pi G t_{nn}^\Phi = G_{AB}[(D_\alpha\Phi^A)(D^\alpha\Phi^B) - (D_n\Phi)^2] \quad (5.4)$$

should have some positive contributions other than the  $(D_nU)^2$  - term, which is zero anyway in the presence of a spacetime reflection symmetry across  $S$ . Thus there should be a *pressure* rather than a *tension* across the surface  $S$ .

In fact we can regard the  $U$  contribution as a purely gravitational contribution to the effective stress tensor. The formula

$$\int_S \mathcal{R} = 2 \int_S K = 0 \quad (5.5)$$

is the statement that the total stresses (gravitational and matter) on the surface  $S$  should vanish. As noticed by Maxwell, gravitational field lines are in compression and exert a tension transverse to the field lines, the opposite behaviour to that of the electric field lines.

If  $S$  were not totally geodesic, it seems that one gets a contribution quadratic in the extrinsic curvature of  $S$  which is like a *bending energy*.

**5.1. Maximum Tension.** If we don't require that the system be asymptotically flat, but merely that  $S$  have the topology of  $\mathbb{R}^2$ , we have (for an outline of the proof and references see [G93])

$$\int_S K < 2\pi \quad (5.6)$$

which yields an upper bound on the tension [G02]

$$-\int t_{ij} n^i n^j \leq \frac{1}{4G}. \quad (5.7)$$

**5.2. Electrostatic Example.** For this the target space is  $SO(2,1)/SO(2)$ ,

$$G_{AB} d\Phi^A d\Phi^B = dU^2 - e^{-2U} d\phi^2 \quad (5.8)$$

and

$$8\pi G t_{ij}^\Phi = 2[(D_i U)(D_j U) - \frac{1}{2} h_{ij} (DU)^2] - 8\pi G e^{-2U} [(D_i \phi)(D_j \phi) - \frac{1}{2} h_{ij} (D\phi)^2] \quad (5.9)$$

Suppose we have a static system of charged bodies with a reflection isometry across  $S$  under which  $d\phi$  is odd, i.e. we have unlike charges. This is the analogue of the situation considered in [ACS] for solitons and in [KK] for Casimir forces. Then the nonexistence-argument in the Theorem in Sect.2 goes through unchanged. The argument would also go through for Einstein-Maxwell-Dilaton theory.

In the case of like charges, as mentioned in Sect.(3.2), there are in fact equilibrium solutions, namely in the Majumdar-Papapetrou case, where  $dU = \pm \sqrt{4\pi G} e^{-U} d\phi$  (whence  $t_{ij}^\Phi = 0$ ),  $h_{ij}$  is flat and  $D^i(e^{-U} D_i \phi) = 0$ . These solutions describe black holes of equal mass and charge where the gravitational attraction is exactly balanced by electrostatic repulsion.

In the Majumdar-Papapetrou case a boost inside  $SO(2,1)$  gives the Harrison transformation and one may boost the well-known Israel-Khan multi-black hole solution [IK] to give a multi-charged solution which however is non-singular along the axis only in the Majumdar-Papapetrou limit [G80].

## 6. 4+1 DIMENSIONS

Many recent interesting applications of black holes, black rings etc have involved simple supergravity in 4+1 spacetime dimensions. The bosonic sector of this theory is essentially Einstein-Maxwell theory supplemented with a crucial Chern-Simons term whose coefficient is dictated by supersymmetry. The equations for stationary metrics may be derived from an action principle given in [LMP].

The Lagrangian 5-form for the five-dimensional theory is

$$R\eta - \frac{1}{2} *F \wedge F + \frac{1}{\sqrt{27}} F \wedge F \wedge A, \quad (6.1)$$

where  $\eta$  is the volume 5-form and  $F = dA$ , and  $A$  is the electromagnetic potential.<sup>5</sup>

One makes the ansatz

$$ds^2 = -e^{-2\phi} (dt + \omega)^2 + e^\phi \gamma_{ij} dx^i dx^j \quad (6.2)$$

$$A = \tilde{A} + \sqrt{3}\chi(dt + \omega). \quad (6.3)$$

The Lagrangian 4-form in 4 dimensions is

$$\tilde{R}\tilde{\eta} - \frac{3}{2} *_\gamma d\phi \wedge d\phi + \frac{3}{2} e^{2\phi} *_\gamma d\chi \wedge d\chi + \frac{1}{2} e^{-3\phi} *_\gamma F \wedge F - \frac{1}{2} e^{-\phi} *_\gamma \tilde{F} \wedge \tilde{F} + \chi d\tilde{A} \wedge d\tilde{A}, \quad (6.4)$$

where

$$F = d\omega, \quad \tilde{F} = d\tilde{A} + \sqrt{3}\chi d\omega. \quad (6.5)$$

Setting  $\chi = 0 = \tilde{A}$  we obtain the purely gravitational theory.  $\phi$  is the Newtonian potential and  $F$  the gravito-magnetic field. The electrostatic potential is  $\chi$  and magnetic field is  $\tilde{F}$ . If one evaluates the effective stress tensor one sees that the last term in (6.4) does not contribute since it is “topological”, i.e. it may be written without any metric. Thus the stress tensors for  $\chi$  and  $\tilde{F}$  are identical in structure but opposite in sign to those for  $\phi$  and  $F$ .

It is the structure of the term involving  $F_{ij}$  which is of principal interest. Let us define

$$t_{ij}^{\text{spin}} = F_{ik} F_{jl} \gamma^{kl} - \frac{1}{4} \gamma_{ij} F_{kl} F_{rs} \gamma^{kr} \gamma^{ls}. \quad (6.6)$$

The scalar curvature of the totally geodesic surface  $S$  contains a term proportional to

$$+ 2t_{nn}^{\text{spin}}. \quad (6.7)$$

Defining

$$\mathcal{E}_\alpha = F_{n\alpha}, \quad \mathcal{B}_3 = F_{12}, \quad \text{etc.} \quad (6.8)$$

one has

$$t_{nn}^{\text{spin}} = \frac{1}{2} \sum_{\alpha=1,2,3} (\mathcal{E}_\alpha \mathcal{E}^\alpha - \mathcal{B}_\alpha \mathcal{B}^\alpha) \quad (6.9)$$

This is positive or negative depending upon whether takes the case in which  $F_{ij}$  is even or odd under reflection in  $S$ .

The resultant pattern of attractions and repulsions is presumably consistent with the detailed models discussed in [HRZ].

## 7. CONCLUSION

In this paper, extending previous work two of us [BSc], we have studied the stationary equilibrium of a system of gravitating bodies in General Relativity, possibility coupled to one or more electromagnetic and scalar fields, which is reflection symmetric about a surface which does not intersect the sources. In the case that the reflection symmetry acts so that that the bodies on one side have the opposite charges and angular momenta to those on the other side, we were able to exclude such a possibility.

<sup>5</sup> This implies that the graviphoton equation of motion is  $\nabla_\nu F^{\mu\nu} + 1/(4\sqrt{3})\epsilon^{\mu\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} = 0$ .

Our principal innovation was to make use of a suitable conformal rescaling of the spatial metric which allowed us to express the necessary condition for equilibrium in terms of a stress tensor for the gravitational, gravito-magnetic fields and any possible scalars or vector fields. The necessary condition resembles closely the elementary requirement for any equilibrium in flat spacetime, that the integrated total stress across any surface must vanish. We have applied our methods to stationary  $(n+1)$  dimensional vacuum spacetimes for arbitrary  $n \geq 3$ . We have also applied them to Einstein Maxwell theory in  $3+1$  dimensions and supergravity theories in  $4+1$  dimensions as well as to various theories in  $3+1$  which arise in super gravity and Kaluza-Klein theories.

An interesting question for further study is whether our methods may be applied to the equilibrium of various extended objects (“p-branes”) which arise in String/M-Theory.

A bigger challenge would be to extend the methods to encompass a cosmological constant, either positive or negative. In the case of negative cosmological constant this might be possible using the various extensions of the positive mass theorem to asymptotically anti-de-Sitter spacetimes. In the case of positive cosmological constant it is less obvious how to proceed.

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